Universality of the Tearing Phase in Matrix Models

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Abstract. The spontaneous symmetry breaking associated to the tearing of a random surface, where large dynamical holes fill the surface, was recently analized obtaining a non-universal critical exponent on a border phase. Here the issue of universality is explained by an independent analysis. The one hole sector of the model is useful to manifest the origin of the (limited) non-universal behaviour, that is the existence of two inequivalent critical points.

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Some years ago, Kazakov analyzed an interesting matrix model describing random surfaces with dynamical holes [1]. He showed that the continuum limit of the model has three phases: a "perturbative" one where small holes do not alter the geometric properties of the random surface; a "tearing phase", where the surface is formed by thin strips surrounding large holes of diverging average length, and a third phase separating the above two where both holes and strips are large and competing.

The model was analyzed mainly as a solvable model of open strings embedded in zero or in one dimension [1,2,3]. Matrix models in reduced dimensions provide interesting statistical models and are suggestive of critical behaviours that may occur in the non perturbative analysis of quantum field theory in more realistic dimension of space time.

Due to the very interesting features of the tearing transition, with the aim of testing the universality of its critical exponents, a matrix model very similar both to Kazakov's model [1] and to the $O(n)$ matrix model [4,5] was recently analyzed [6]. The qualitative description of the critical behaviour turned to be equivalent, with the same critical exponents in the two critical phases (small holes phase or perturbative phase, and the tearing phase) but a different one on the border phase. This discrepancy was unexpected and it deserves deeper understanding, provided by the present letter.

In the first part of this work we study the model by the method of orthogonal polynomials and find the scaling laws which characterize the scaling behaviour of the holes on the random surface in our model. In the second part we restrict the model to the "one -hole" sector: we investigate the case of a "static" loop interacting with the random surface. A first-order transition is found, corresponding to the hole filling the surface. This first-order transition is the memory of the tearing phase in the one-hole sector.

The partition function of the model is

$$Z_N(L,g,z) = \int \mathcal{D}M \exp\{-N\mathrm{Tr}[V(M) + L\log(1 - 2zM)]\} \qquad (1)$$

where M is an hermitian $N \times N$ matrix, $g > 0$, $z > 0$ and the potential is

$$V(M) = \frac{1}{2}M^2 + \frac{g}{3}M^3 \qquad (2)$$

We refer the reader to ref.[6] for the notations and an analysis of the relations between this model and Kazakov's model as well as the $O(n)$ model.

Let us introduce $\lambda = g^2$, $\gamma = g^2 L$ and $\mu = \frac{2z}{g}$. The fugacity of the number of holes on the surface is γ, while μ turns out to be the effective fugacity of the total perimeter of the holes. After a rescaling $\phi = gM$ the partition function may be written as

$$Z = \int d\phi \exp -\mathrm{tr}\frac{N}{\lambda}\left(\frac{1}{2}\phi^2 + \frac{1}{3}\phi^3 + \gamma\ln(1 - \mu\phi)\right) = \int d\phi \exp -\mathrm{tr}\frac{N}{\lambda}\mathcal{V}(\phi) \qquad (3)$$

The matrix model can be studied by the standard method of orthogonal polynomials [7]. Let us introduce the set $\langle \varphi | n \rangle = P_n(\varphi)$ of polynomials orthonormal with respect to the measure $d\mu = d\varphi e^{-\frac{N}{\lambda}\mathcal{V}(\varphi)}$:

$$\langle m | n \rangle = \int d\mu P_m(\varphi) P_n(\varphi) = \delta_{mn} \qquad (4)$$

The coordinate operator $\hat{\varphi} : g(\varphi) \rightarrow \varphi g(\varphi)$ has the following matrix elements:

$$\langle m | \hat{\varphi} | n \rangle = \sqrt{R_m}\delta_{m,n+1} + S_n \delta_{m,n} + \sqrt{R_n}\delta_{m,n-1} \qquad (5)$$

Then

$$Z = N! C^N \prod_{i=1}^{N-1} R_i^{N-i} \qquad (6)$$

where the constant C is the normalization of the measure: $C = \int d\mu$.

The coefficients R_n and S_n are determined by the "equations of motion":

$$\langle n | \mathcal{V}'(\hat{\varphi}) | n \rangle = 0 \qquad (7a)$$

$$\langle n-1 | \mathcal{V}'(\hat{\varphi}) | n \rangle = \frac{n\lambda}{N\sqrt{R_n}} \qquad (7b)$$

which for our potential have the form:

$$0 = S_n + S_n^2 + R_{n+1} + R_n - \mu\gamma \langle n | \frac{1}{1 - \mu\hat{\varphi}} | n \rangle \qquad (8a)$$

$$\frac{n\lambda}{N\sqrt{R_n}} = \sqrt{R_n}(1 + S_n + S_{n-1}) - \mu\gamma \langle n-1 | \frac{1}{1 - \mu\hat{\varphi}} | n \rangle \qquad (8b)$$

The operator $(1 - \mu\hat{\varphi})^{-1}$ is the resolvent of a random motion on the lattice $\mathbf{N}$. In order to perform the planar limit $N \rightarrow \infty$ it is convenient to introduce the conjugate operators $\hat{l}$ and $\hat{\theta}$ [8], defined by

$$\hat{l}|n\rangle = \frac{n}{N}|n\rangle \ , \ e^{\pm i\hat{\theta}}|n\rangle = |n \pm 1\rangle \qquad (9)$$

The operator $\hat{\varphi}$ can be expressed as

$$\hat{\varphi} = \sqrt{R(\hat{l})}e^{i\hat{\theta}} + S(\hat{l}) + e^{-i\hat{\theta}}\sqrt{R(\hat{l})} \qquad (10)$$

and in the $\theta - basis$, $|\theta\rangle = \frac{1}{\sqrt{2\pi}}\sum_n e^{in\theta}|n\rangle$, $\hat{l}$ acts as a derivative.

In the large N limit $\hat{l}$ commutes with $\hat{\theta}$ and can be taken equal to the identity. The operator $\hat{\varphi}$ simplifies to:

$$\hat{\varphi} = 2\sqrt{R}cos\theta + S \qquad (11)$$

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having assumed the limits

$$R = \lim_{n \to \infty} R_n , \quad S = \lim_{n \to \infty} S_n$$

In the planar limit we have

$$\langle n | (1 - \mu \hat{\varphi})^{-1} | n \rangle \to \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{1}{1 - \mu \varphi(\theta)} = \frac{1}{\sqrt{(1 - \mu S)^2 - 4\mu^2 R}} \qquad (12)$$

and

$$\langle n - 1 | (1 - \mu \hat{\varphi})^{-1} | n \rangle \to \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{e^{i\theta}}{1 - \mu \varphi(\theta)} = \frac{1}{2\mu \sqrt{R}} \left(\frac{1 - \mu S}{\sqrt{(1 - \mu S)^2 - 4\mu^2 R}} - 1 \right) \quad (13)$$

The equations of motion in the planar limit read

$$0 = S + S^2 + 2R - \mu \gamma G^{-1}(R, S, \mu) \qquad (14a)$$

$$\lambda = R + 2RS + \frac{\gamma}{2} \left[1 - (1 - \mu S) G^{-1}(R, S, \mu) \right] \qquad (14b)$$

where we denote $G(R, S, \mu) = \sqrt{(1 - \mu S)^2 - 4\mu^2 R}$; eqs.(14) correspond to eqs.(4.8) in ref.[6] with the identifications

$$S = \sigma , \quad R = \frac{\delta^2}{4} , \quad \mu = \frac{1}{\tau}$$

The above equations provide $S = S(\mu, \gamma, \lambda)$ and $R = R(\mu, \gamma, \lambda)$. The continuum limit of the system corresponds to a critical surface $f(\mu, \gamma, \lambda) = 0$ in the three-dimensional parameter space spanned by the variables μ, γ and λ. We can ensure critical behaviour by imposing the following scaling:

$$S = S_0 + S_1 a , \quad R = R_0 + R_1 a$$

$$\lambda = \lambda_0 + \Lambda a^\ell , \quad \gamma = \gamma_0 + \Gamma a^k \qquad (15)$$

with $\ell > 1$ and $k > 1$, where a is a cut-off vanishing in the continuum limit; indeed eqs.(15) imply $\frac{\partial S}{\partial \lambda} = \infty = \frac{\partial R}{\partial \lambda}$.

The condition $G(\mu, S_0, R_0) \neq 0$ characterizes the perturbative phase. Inserting the scaling laws (15) in eqs.(14) and requiring non-trivial solutions for S_1 and R_1 leads to the equation:

$$4R_0(1 + 3S_0 - \mu^{-1})^2 = [(1 + 2S_0)(\mu^{-1} - S_0) - S_0(1 + S_0) - 6R_0]^2 \qquad (16)$$

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fully equivalent to the critical equation (4.9) in ref.[6]. As in Kazakov's model [1], the analysis of the critical behaviour is simplified by considering the values of γ with $\gamma_0 = 0$. The critical values are then $S_0 = \frac{-3+\sqrt{3}}{6}$, $R_0 = \frac{1}{12}$, $\lambda_0 = \frac{1}{12\sqrt{3}}$; the consistent value for the exponents ℓ and k is 2. This is the "small holes" phase [1,6].

Let us now consider the non-perturbative phase.

When $G(\mu, R_0, S_0)$ tends to zero as γ vanishes a new critical behaviour arises: the phenomenon of spontaneous tearing discussed in ref.[1,6]. Inserting the scaling laws (15) ,with $\gamma_0 = 0$ and the condition $G(\mu, R_0, S_0) = 0$, in eqs.(14) (we observe that in the non-perturbative phase ℓ is not supposed to be greater than one because criticality is ensured by the vanishing of G), we obtain:

$$R_0(\mu) = -\frac{1}{2}S_0(S_0 + 1) \; , \; S_0(\mu) = \frac{1}{3\mu}\left(1 - \mu + \sqrt{(1-\mu)^2 - 3}\right)$$

$$\lambda_0(\mu) = -\frac{1}{2}S_0(1 + S_0)(1 + 2S_0) \tag{17}$$

and the equation

$$2\mu^2\Lambda = (\mu + 3\mu S_0 - 1)[2R_1 - \frac{S_1}{\mu}(1 - \mu S_0)] \tag{18}$$

The square root in (17) implies $\mu \geq \mu_c = (1 + \sqrt{3})$ for the non-perturbative phase, and the consistent values for ℓ and k are respectively 1 and 3/2.

The case $\mu = \mu_c$ (critical tearing) has to be investigated separately, since eq.(18) implies $\Lambda = 0$ in this limit. Note that eqs.(14) may be rewritten as

$$R = \frac{\lambda - \frac{\gamma}{2} + \frac{S}{2\mu}(1 + S)(1 - \mu S)}{1 + 3S - \mu^{-1}} \tag{19a}$$

$$0 = S(1 + S)(1 + 2S) + 2\lambda - \gamma - \mu\gamma(1 + 3S - \mu^{-1})G^{-1} \tag{19b}$$

with $1 + 3S_0 - \mu_c^{-1} = 0$. It is straightforward to check that the scaling law compatible with eqs.(19) when $\mu \to \mu_c$ is

$$R = R_0 + R_1 a \; , \; S = S_0 + S_1 a \; , \; \mu = \mu_c - Ma$$

$$\lambda = \lambda_0 + \Lambda a^{3/2} \; , \; \gamma = \Gamma a^{3/2} \tag{20}$$

to be compared with the corresponding law in Kazakov's model [2]:

$$R = R_0 + R_1 a \; , \; \mu = \mu_c - Ma$$

$$\lambda = \lambda_0 + \Lambda a^2 \; , \; \gamma = \Gamma a^{5/2} \tag{21}$$

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It follows that in our model the dynamical holes exhibit, in the intermediate phase, a different scaling behaviour with respect to Kazakov's model. Indeed the typical area of the surface diverges at criticality as $\frac{1}{\lambda_c - \lambda}$, while the total perimeter of the holes on the surface diverges as $\frac{1}{\mu_c - \mu}$. Then in the intermediate phase (critical tearing) the scaling laws (20) imply for our model that the "length" of the holes scales as the area to the power of 2/3, while eqs.(21) imply for Kazakov's model that in the intermediate phase the length of the holes scales as the square root of the area.

It is interesting to observe that if we defined our model with the potential

$$V_1(M) = \frac{1}{2}M^2 - \frac{g}{3}M^3 \ , \ g > 0 \tag{22}$$

instead of (2), then the equations corresponding to (19) would be:

$$R = \frac{\lambda - \frac{\gamma}{2} + \frac{S}{2\mu}(1 - S)(1 - \mu S)}{1 - 3S + \mu^{-1}} \tag{23a}$$

$$0 = S(1 - S)(1 - 2S) - 2\lambda + \gamma - \mu\gamma(1 - 3S + \mu^{-1})G^{-1} \tag{23b}$$

with the critical values $S_0 = \frac{3 - \sqrt{3}}{6}$, $R_0 = \frac{1}{12}$ and $\mu_c = 3 - \sqrt{3}$ the positive solution of the equation

$$(1 - \mu_c S_0)^2 - 4\mu_c^2 R_0 = 0$$

In this case $1 - 3S_0 + \mu_c^{-1} \neq 0$ and eqs.(23) admit a scaling law completely analogous to (21) implying the same scaling behaviour for the holes as in Kazakov's model even in the intermediate phase.

Let us explain this point. The one matrix model

$$V(M) = \frac{1}{2}M^2 + \frac{g}{3}M^3$$

is invariant under $g \to -g$ and $M \to -M$, so it has two critical points, g^* and $-g^*$. The two critical points are equivalent for the pure cubic model and they both describe pure gravity. When the random surface is coupled to the holes the two critical points are no more equivalent: if the surface reaches the continuum limit by sending g to g^* then the holes always have the same scaling behaviour as in Kazakov's model, while sending g to $-g^*$ the holes have a different scaling behaviour in the intermediate phase.

The "anomalous" scaling behaviour of the dynamical holes in the intermediate phase is connected with the following feature of our model in the one-hole sector: the absence of the dilute phase for the single static hole interacting with the random surface.

The one-hole sector is obtained, as explained in ref.[6], considering the formal Taylor expansion in L of the free energy of model (1)

$$E(L, g, z) = -\lim_{N \to \infty} \frac{1}{N^2} \ln Z_N = \sum_{k=0}^{\infty} L^k E_k(g, z) \tag{24}$$

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The term E_1 is the generator of planar connected graphs with one hole. In terms of the density of eigenvalues $\rho_3(\lambda)$ of the pure one matrix cubic model [9] E_1 is given by the following integral:

$$E_1(g, z) = \int_{a_0}^{b_0} d\lambda \rho_3(\lambda) \ln (1 - 2z\lambda) \tag{25}$$

The singularity $g_c(\tau)$ of E_1 (where $\tau = \frac{g}{2z}$) yields the density of free energy in the thermodynamic limit [6]:

$$f = \ln g_c(\tau)$$

by means of which we can evaluate the average length of the perimeter of the hole per unit area $\mathcal{L} = \frac{\partial f}{\partial \ln \tau}$.

The density of length of the hole $\mathcal{L}$ plays here the role of an order parameter: $\mathcal{L} = 0$ corresponds to a "confined" polymer having a finite perimeter, $\mathcal{L} \neq 0$ corresponds to a polymer with infinite length which is dense on the surface (indeed the hole boundary is a fractal).

The explicit expression of E_1 has been evaluated in [6]. Its singularity may arise from the singularity of ρ_3 with respect to g or from the vanishing of the argument of the log in eq.(25) at $\lambda = b_0$, i.e. condition $\frac{1}{2z} = b_0$.

In the range $\tau > \tau_c = \frac{1}{2}(\sqrt{3} - 1)$ the singularity of E_1 is given by the singularity of ρ_0 and $\mathcal{L} = 0$:

$$g_c^2 = \frac{1}{12\sqrt{3}} \ , \ \tau > \tau_c \tag{26}$$

In the range $0 < \tau < \tau_c$ the singularity is due to the condition $\frac{1}{2z} = b_0$ and the parametric expression of g_c is

$$2g_c^2 + \sigma(1 + \sigma)(1 + 2\sigma) = 0$$

$$\tau = \sigma + \sqrt{-2\sigma(1 + \sigma)} \ , \ 0 < \tau < \tau_c \tag{27}$$

which imply $\mathcal{L} \neq 0$, i.e. the hole is dense on the surface. In fig.(1a) g_c versus τ is plotted. We see that $g_c(\tau)$ is continous at τ_c but its first derivate (proportional to $\mathcal{L}$) is not. Hence a first-order transition occurs with the absence of the dilute phase for the polymer [5].

The one-hole sector in Kazakov's model is defined by

$$E_1 = \int_{-a}^{a} d\lambda \rho_4(\lambda) \ln (1 - z^2\lambda^2) \tag{28}$$

with $\rho_4(\lambda)$ being the density for the pure quartic model [9]. By setting $\tau = \frac{g}{z^2}$, one easily finds

$$g_c(\tau) = \frac{\tau}{4} - \frac{3\tau^2}{16} \ , \ 0 < \tau < 2/3$$

$$g_c(\tau) = \frac{1}{12} \ , \ \tau > 2/3 \tag{29}$$

The phase transition is second order. In fig.(1b) the critical coupling of Kazakov's model in the one-hole sector is plotted versus the fugacity. In this case a dilute phase for the polymer is found, corresponding to the critical fugacity, where the length of the polymer scales as the square root of the area of the surface.

Let us now compare our results for the single hole with the analysis of self avoiding walks on random surface [5]. The case in which two random walks tied together at their ends live on a random trivalent lattice is equivalent to our E_1 model with the difference that the two SAWs form a loop but not one hole. In fig.(1c) the critical coupling versus the fugacity is plotted in this case, showing that a second order transition occurs. The transition point corresponds to the dilute phase. It is interesting to observe that the model with $V_1(M)$, eq.(22), instead of (2) in model (1) has in the one-hole sector a critical curve $g_c(\tau)$ for the single hole which is exactly the same as the one plotted in fig.(1c) for the two SAWs. Conversely changing the sign of the coupling constant in the two SAWs model yields exactly the critical curve in fig.(1a) and implies the absence of the dilute phase.

These results seem to suggest the following relation between a model of dynamical holes on a random surface and the corresponding one-hole sector: the transition in the one-hole sector is second-order if and only if the dynamical holes have at critical tearing the "standard" scaling behaviour (with the length of the holes scaling as the square root of the area of the surface).

Let us summarize the main results of this letter:

(1) We exhibit the scaling behaviours proper to the continuum limit for the three critical phases and we confirm the critical exponents found in [6].

(2) The existence of two inequivalent critical points for the model of self avoiding walks on random surfaces is here shown. The two points occur for opposite values of the cubic coupling. They correspond, in models with dynamical loops, to two inequivalent critical lines. One was described in the paper [6], the other, also discussed here, yields the same exponents as the critical line in Kazakov 's model.

Figure Captions

Fig.1 Critical coupling $g_c(\tau)$ of the one-hole sector (τ is the inverse of the fugacity of the length of the polymer) for:

(a) Our model, see eqs.(26,27).

(b) Kazakov's model, see eq.(29).

(c) Self-avoiding-walks, ref.[5], or our model with potential $V_1(M)$, eq.(22).

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